ECE 874 **Test 1 In-Class Component** Name ANS

(50 points total)

1. (5 pts) Given the following system:

$$\dot{x}_1 = -2x_2$$

$$\dot{x}_2 = x_1 - \frac{1}{3}x_1^3 - 3x_2$$

$0 = -2x_2 \Rightarrow x_2 = 0$

$0 = x_1 - \frac{1}{3}x_1^3 - (0)$

$x_1(1 - \frac{1}{3}x_1^2) = 0 \Rightarrow x_1 = 0$

$x_1 = \pm\sqrt{3}$

a) Find all equilibrium points.

$(0,0), (\sqrt{3},0), (-\sqrt{3},0)$

b) Show that the *origin* is stable. The Jacobian $\frac{\partial f}{\partial x}(x_e) = \begin{bmatrix} 0 & -2 \\ 1 - x_1^2 & -3 \end{bmatrix}\Bigg|_{x=x_e}$ where x_e are the equilibra.

Linearization

$A = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$

find eigen values

$|\lambda I - A| = 0$

$\left|\begin{bmatrix} \lambda & 2 \\ -1 & \lambda+3 \end{bmatrix}\right| = \lambda^2 + 3\lambda + 2$

$= (\lambda+2)(\lambda+1)$

$\lambda = -2 \quad \lambda = -1$

$Re(\lambda_i) < 0 \Rightarrow$ stable by theorem

2. (5 pts) Identify the following functions as PD, PSD, ND, NSD, none:

a) $V(x) = x_1^2 + x_2^2$ with $x = [x_1 \quad x_2 \quad x_3]^T$ is PD, PSD, ND, NSD, none (circle one)

b) $V(x) = x_1^2 + x_2^2$ with $x = [x_1 \quad x_2]^T$ is PD, PSD, ND, NSD, none (circle one)

c) $V(x) = -x_1\sin(x_1) - x_2^2$ with $x = [x_1 \quad x_2]^T$ is PD, PSD, ND, NSD, none (circle one)

d) $V(x) = -x_1^2 - 2x_1x_2 - x_2^2 = -(x_1 + x_2)^2$ with $x = [x_1 \quad x_2]^T$ is PD, PSD, ND, NSD, none (circle one)

e) $V(x) = -x_1^2 + x_2^2$ with $x = [x_1 \quad x_2]^T$ is PD, PSD, ND, NSD, none (circle one)

3. (5 pts) In designing the control u in the following system, which terms *must* be canceled by the controller.

a) Circle the terms that you *must* cancel with your control to ensure Global Asymptotic Stability.

$$\dot{x}_1 = -x_1 - x_1^2 - x_1^{20} + x^3 + x^4 - \sin(x_1) - \cos(x_1) + u$$

b) Circle the terms that you *must* cancel with your control to ensure Local Asymptotic Stability.

$$\dot{x}_1 = -x_1 - x_1^2 - x_1^3 + x^4 + x^5 - \sin(x_1) - \cos(x_1) + \tan(x_1) + u$$

a is a positive constant ($a > 0$) that you should specify as part of your design

6. (10 pts) Design a nonlinear feedback control law, u, that makes the origin globally asymptotically stable. (Don't make this harder than it is)

$$\dot{x}_1 = -3x_1 + 2ax_1x_2^2 + u$$
$$\dot{x}_2 = -ax_2^3 - (2a-1)x_2$$

$$V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$$
$$\dot{V} = x_1(-3x_1 + 2ax_1x_2^2 + u) + x_2(-ax_2^3 - (2a-1)x_2)$$
$$= -3x_1^2 + 2ax_1^2x_2^2 + ux_1$$
$$-ax_2^4 - (2a-1)x_2^2$$

specify: $a = \frac{1}{2}$ or $a > \frac{1}{2}$

$$u = -2a x_1 x_2^2$$
$$\dot{V} = -3x_1^2 - a x_2^4 - (2a-1)x_2^2$$

≥ 0 depending on your choice of "a"

$V(0) = 0$
V is PD
$\dot{V}$ is ND $\Rightarrow x_1, x_2 \to 0$
$u \to 0$
$\dot{x}_1, \dot{x}_2 \to 0$

7. (10 pts) A system is modeled as:

$$\dot{x}_1 = -x_1 + x_2 - \sin(x_1)$$
$$\dot{x}_2 = -x_1 + \cos(x_2) + u$$

A backstepping control has been designed as

$$u = -x_1\cos(x_1) + x_2\cos(x_1) - \sin(x_1)\cos(x_1) - \cos(x_2) - \eta_2$$
$$\eta_2 = x_2 - x_{2d}$$
$$x_{2d} = \sin(x_1)$$

via a Lyapunov analysis:

$$V = x_1^2 + \eta_2^2$$
$$\dot{V} = -x_1^2 - \eta_2^2$$

Finish the design by showing that all signals are bounded.

$V(0) = 0$, V PD, $\dot{V}$ ND $\Rightarrow x_1, \eta_2 \to 0$

$x_1 \to 0 \Rightarrow x_{2d} = \sin(x_1) \to 0$

$\eta_2, x_{2d} \to 0 \Rightarrow x_2 = \eta_2 + x_{2d} \to 0$

$x_1, x_2, \eta_2 \to 0 \Rightarrow u \to -1$

$x_1, x_2 \to 0 \Rightarrow \dot{x}_1 \to 0$

$x_1, x_2 \to 0, u \to 1 \Rightarrow \dot{x}_2 = -0 + \cos(0) - 1 = 0$

4. (5 pts) Use the Lyapunov function candidate $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ to show that the origin of the following system is GAS.

$$\dot{x}_1 = -x_1^3 - 2x_2$$
$$\dot{x}_2 = 2x_1 - x_2$$

V is PD, $V(0) = 0$

$$\dot{V} = x_1 \dot{x}_1 + x_2 \dot{x}_2$$
$$= x_1(-x_1^3 - 2x_2) + x_2(2x_1 - x_2)$$
$$= -x_1^4 - \cancel{2x_1x_2} + \cancel{2x_1x_2} - x_2^2$$
$$= -x_1^4 - x_2^2$$

$\Rightarrow \dot{V}$ is ND

$V(0) = 0$, V PD, $\dot{V}$ ND $\Rightarrow$ GAS

5. (10pts) Use the Lyapunov function candidate $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ to show that the origin of the following system is locally AS and give an estimate of the region of attraction.

$$\dot{x}_1 = -1.5x_1 + 3x_1x_2$$
$$\dot{x}_2 = [illegible] - x_2$$

$V(0) = 0$, V is PD

$$\dot{V} = -1.5x_1^2 + 3x_1^2x_2 - x_2^2$$
$$= (-1.5 + 3x_2)x_1^2 - x_2^2$$

Need $-1.5 + 3x_2 < 0$

$x_2 < 0.5$

$D = \{x \in \mathbb{R}^2 : x_2 < 0.5\}$

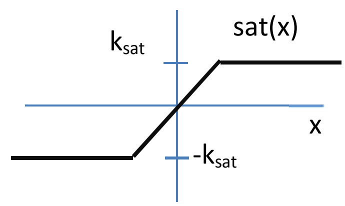
ECE 874 **Test 1 Out-of-Class Component** Name ______________________

(50 points total)

Instructions:

1) The test is due on Monday March 7 in class at 11:15AM.
2) You must show all steps in your solutions
3) Your exam solutions are to be your own work, you are not to give or receive assistance of any kind on this exam.

1. (15 pts) Given the system



$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = \tfrac{1}{2}(-x_2 + \sin(x_1) - sat(x_2))$$

a) Find the equilibrium points assuming $k_{sat} = \frac{\pi}{2}$

b) Calculate (by hand) the linearization about the equilibirum points and describe their local stability properties assuming $k_{sat} = \frac{\pi}{2}$.

c) Plot the phase-plane portraits for $-6 < x_1 < 6$ and $-2 < x_2 < 2$ for the values k_{sat} =10, 1.5, 0.1

What is the effect of the saturation (and k_{sat}) on the system?

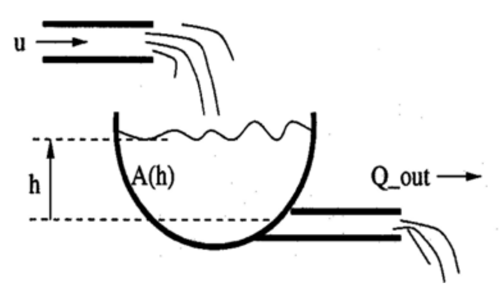
Define the following MATLAB function to implement the saturation.

```
function y=sat(in1,satlimit)
if in1>satlimit
    in1=satlimit;
end
if in1<-satlimit
    in1=-satlimit;
end
y=in1;
```

2. (15 pts)

Consider the problem of controlling the height, h, of fluid in the tank shown in fig With the fluid level starting at some initial height, h_0, the goal is to choose the input flow rate, u, which brings h as close as possible to h_d, the desired height. The tank has a cross-sectional area, A(h), which is a function of the height of the fluid in the tank. A spigot at the bottom of the tank leaks fluid at a rate proportional to the square root of the height of fluid in the tank, $Q_{out} = a\sqrt{2gh}$, where a is a constant of proportionality and g is the acceleration of gravity. Conservation of mass gives the following nonlinear differential equation governing the height of fluid in the tank:

$$A(h)\dot{h} = u - a\sqrt{2gh}$$



a. Assume $A(h) = h^2 + 0.1$ and all parameters are exactly known and h can be measured, design a controller u so that h tracks h_d. Show all work and that all signals are bounded.
b. Simulate the system in Simulink using a=1. Show h(t) on one plot, show u(t) on one plot, show your Lyapunov function and its derivative on one plot.

3. (20 pts)

a. Use the hand-crafted backstepping approach to design a feedback control for the following. Show the stability result for the closed-loop system and that all signals remain bounded

$$\dot{x}_1 = -\cos(x_1) - x_1^3 + x_2 + 3$$

$$\dot{x}_2 = x_1 x_2 - u$$

b. Simulate the system using Simulink. Show states on one plot, show control signals on one plot, and show your Lyapunov function and its derivative on one plot.

1. $\dot{x}_1 = -x_2$

$\dot{x}_2 = \frac{1}{2}(-x_2 + \sin(x_1) - \text{sat}(x_2))$

a.) <u>EQ Point</u>

$\dot{x}_1 = 0 \rightarrow x_2 = 0$

$\dot{x}_2 = \frac{1}{2}(-0 + \sin(x_1) - 0) = 0$

$\sin(x_1) = 0 \Rightarrow x_1 = 0, \pm\pi, \pm 2\pi, \ldots \pm n\pi$

Equilibrium points $(0,0)$, $(0, \pm n\pi)$ $n = 1, \ldots \infty$

b.) $\frac{\partial f_1}{\partial x_1} = 0$; $\frac{\partial f_1}{\partial x_2} = -1$; $\frac{\partial f_2}{\partial x_1} = \frac{1}{2}\cos(x_1)$

$\frac{\partial f_2}{\partial x_2} = -\frac{1}{2} - \frac{1}{2} = -1$

$\text{sat}(x_2) = x_2$ for x_2 below $\pi/2$.

$$A\Big|_{x_e} = \begin{bmatrix} 0 & -1 \\ \frac{1}{2}\cos(x_1) & -1 \end{bmatrix}\Bigg|_{x = x_e}$$

$$A_{(0,0)} = \begin{bmatrix} 0 & -1 \\ \frac{1}{2} & -1 \end{bmatrix} \Rightarrow \lambda^2 + \lambda + \tfrac{1}{2} = 0$$

$\lambda_1 = -.5 - j0.5$

$\lambda_2 = -.5 + j0.5$

$\text{Re}(\lambda_i) < 0 \Rightarrow$ <u>stable</u> at $(0,0)$

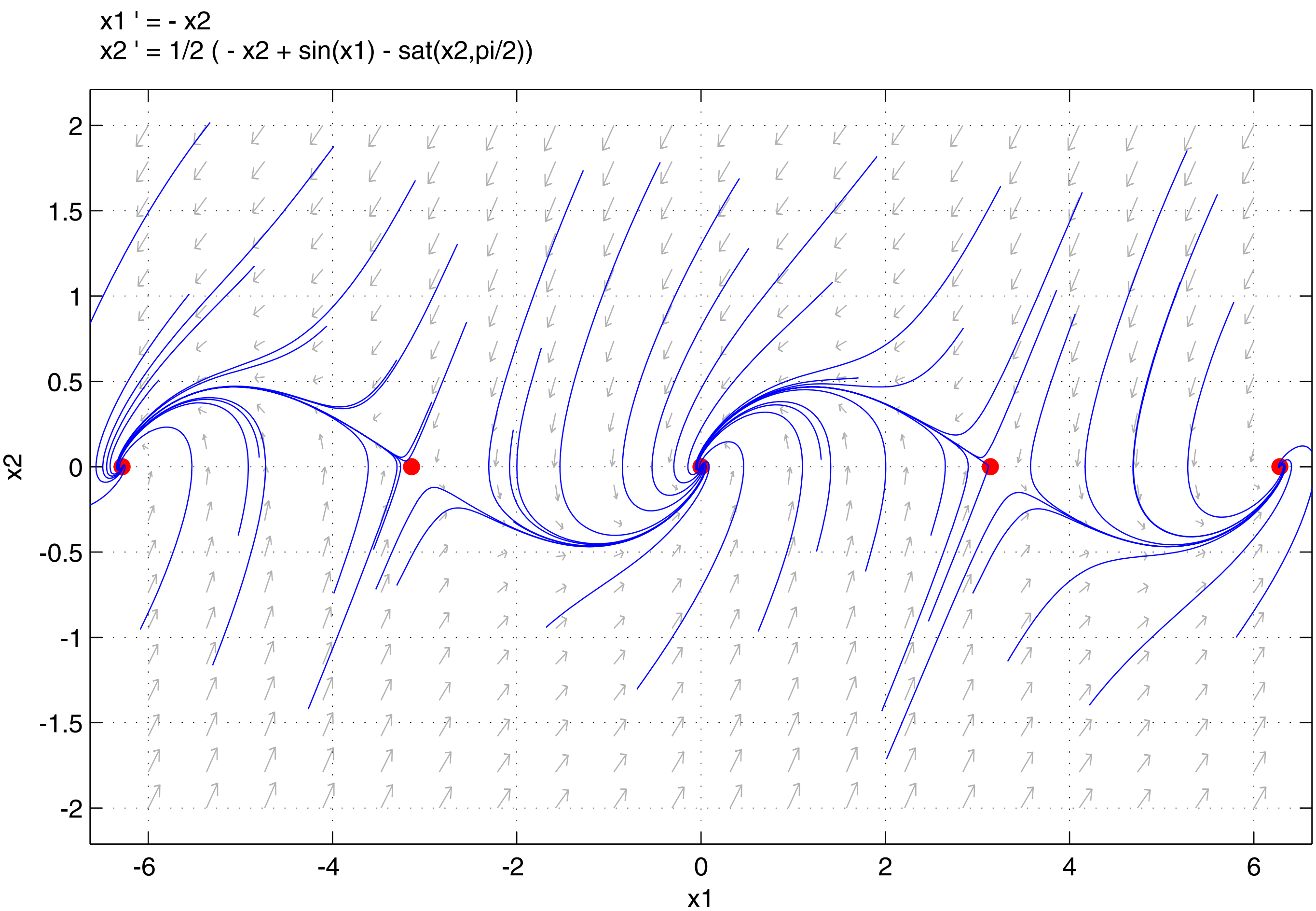
A will be the same for $(0, n\pi)$ where n is an even number. Thus <u>stable</u> at $(0, 2n\pi)$ for $n = 1, \ldots \infty$.

$$A\Big|_{x_e = 2n+1} = \begin{bmatrix} 0 & -1 \\ -\frac{1}{2} & -1 \end{bmatrix} \Rightarrow \lambda^2 + \lambda - \tfrac{1}{2} = 0$$

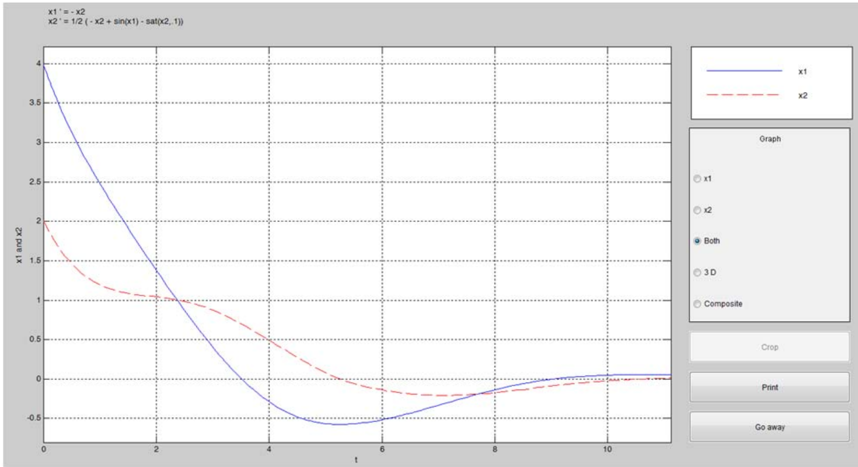
$\lambda_1 = .366$

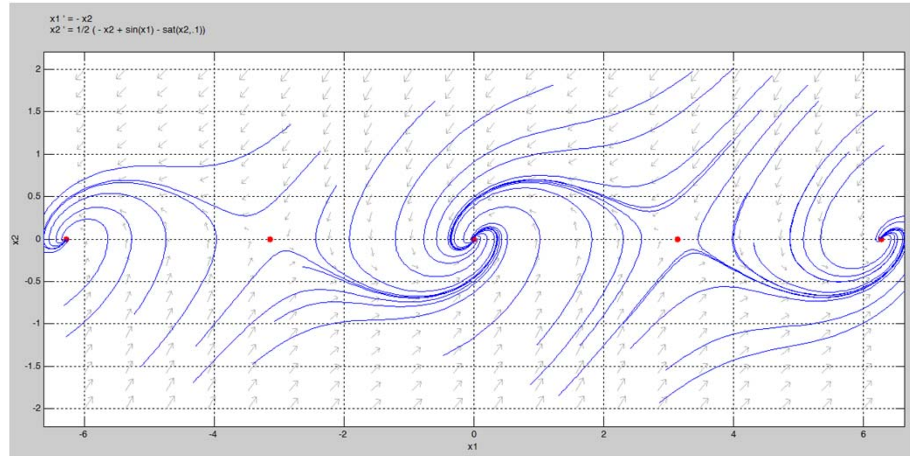
$\lambda_2 = -1.366$ $\Rightarrow$ Saddle point

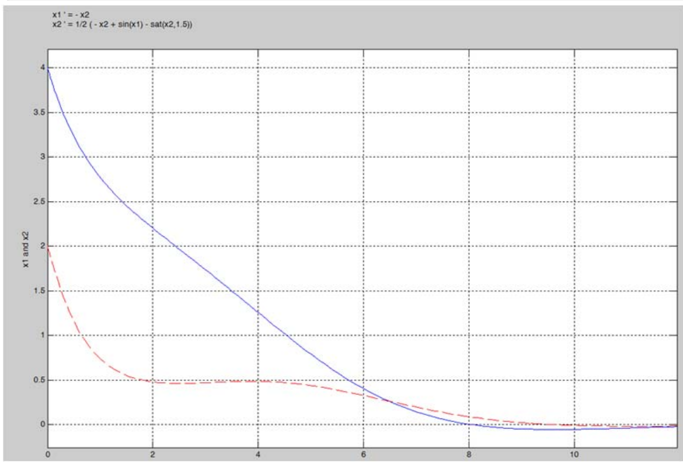
$\Rightarrow$ <u>unstable</u>

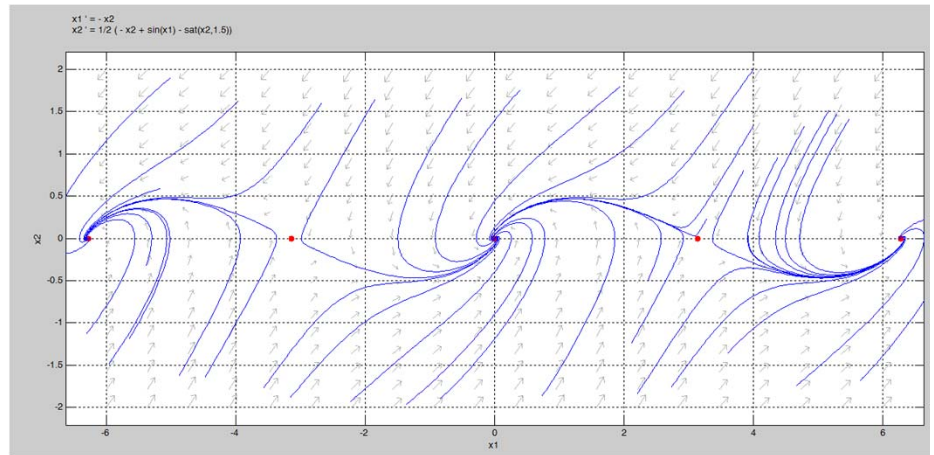


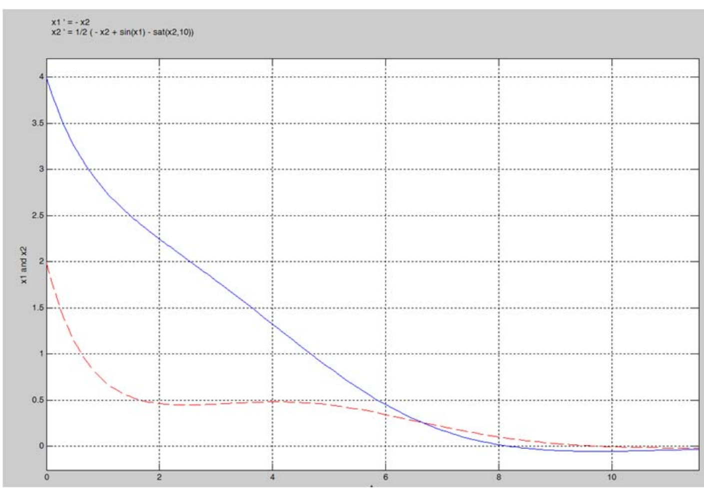
x1 ' = - x2
x2 ' = 1/2 (- x2 + sin(x1) - sat(x2,pi/2))
x1
x2

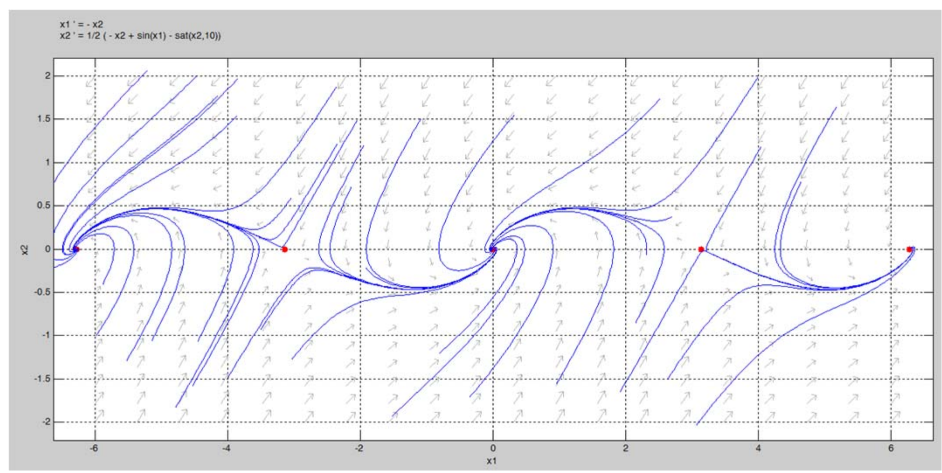

x1 ' = - x2
x2 ' = 1/2 (- x2 + sin(x1) - sat(x2,.1))
x1 and x2
t
x1
x2
Graph
x1
x2
Both
3 D
Composite
Crop
Print
Go away


x1 ' = - x2
x2 ' = 1/2 (- x2 + sin(x1) - sat(x2,.1))
x2
x1


x1 ' = - x2
x2 ' = 1/2 (- x2 + sin(x1) - sat(x2,1.5))
x1 and x2


x1 ' = - x2
x2 ' = 1/2 (- x2 + sin(x1) - sat(x2,1.5))
x2
x1


x1 ' = - x2
x2 ' = 1/2 (- x2 + sin(x1) - sat(x2,10))
x1 and x2


x1 ' = - x2
x2 ' = 1/2 (- x2 + sin(x1) - sat(x2,10))
x2
x1

2. $(h^2+0.1)\dot{h} = u - a\sqrt{gh}$

$$\dot{h} = \frac{1}{(h^2+0.1)}\left(u - a\sqrt{gh}\right)$$

shift system $\bar{h} = h - h_d$

$h = \bar{h} + h_d$, $\dot{h} = \dot{\bar{h}}$

$$\dot{\bar{h}} = \frac{1}{(\bar{h}+h_d)^2+0.1}\left(u - a\sqrt{g(\bar{h}+h_d)}\right)$$

$V = \frac{1}{2}\bar{h}^2$ PD, radially unbounded, $V(0)=0$

$$\dot{V} = \bar{h}\dot{\bar{h}} = \bar{h}\left(\frac{1}{(\bar{h}+h_d)^2+1}\left(u - a\sqrt{g(\bar{h}+h_d)}\right)\right)$$

design $u = a\sqrt{g(\bar{h}+h_d)} - \bar{h}\left[(\bar{h}+h_d)^2+1\right]$

$\dot{V} = -\bar{h}^2$ ND

V is PD, rad unbounded, $V(0)=0$ $\Rightarrow$ $\bar{h} \to 0$

$\dot{V}$ is ND

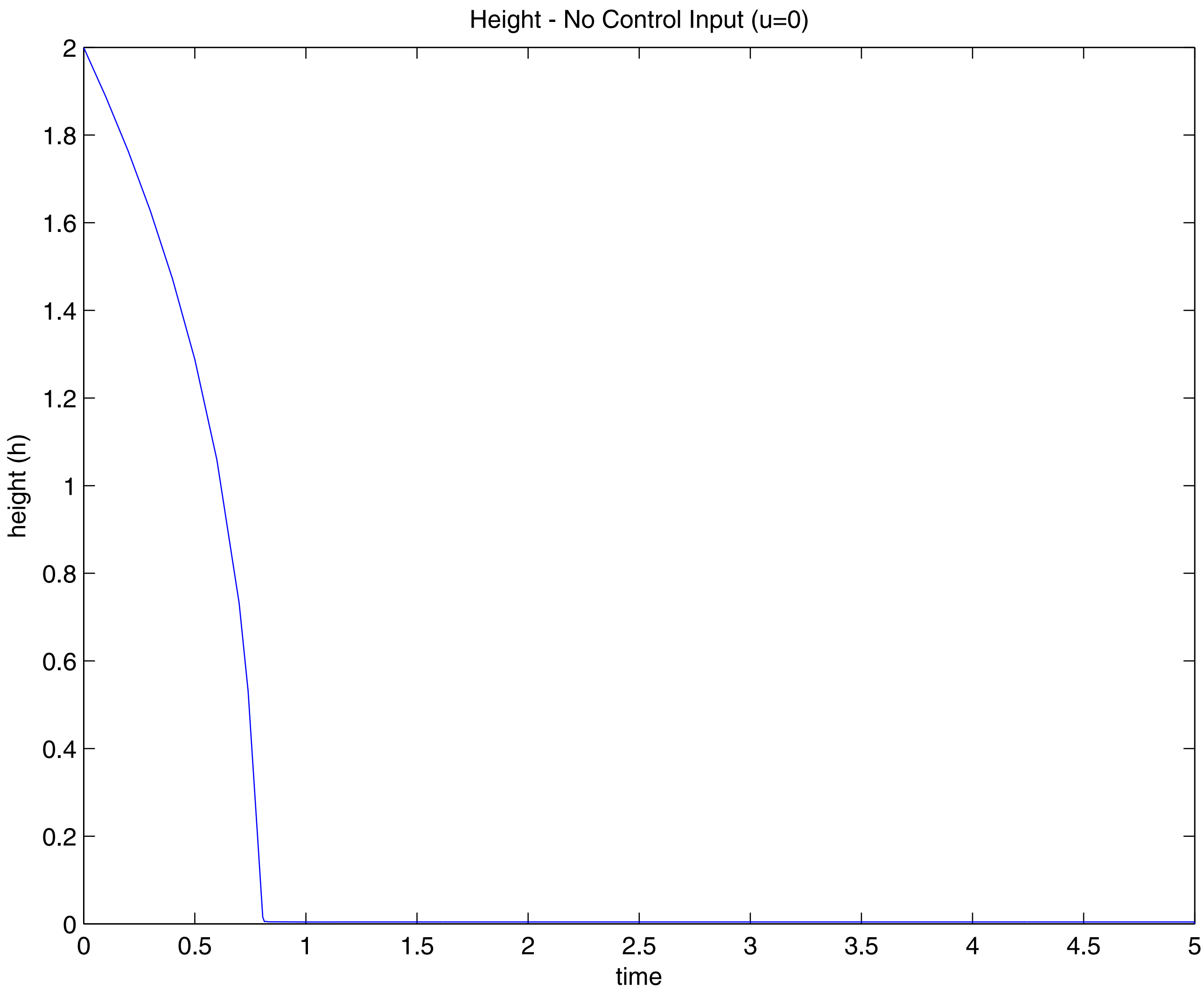
since $\bar{h} \to 0$ & h_d is a bounded constant $\Rightarrow h \to h_d$

$\dot{\bar{h}} = -\bar{h} \to 0$

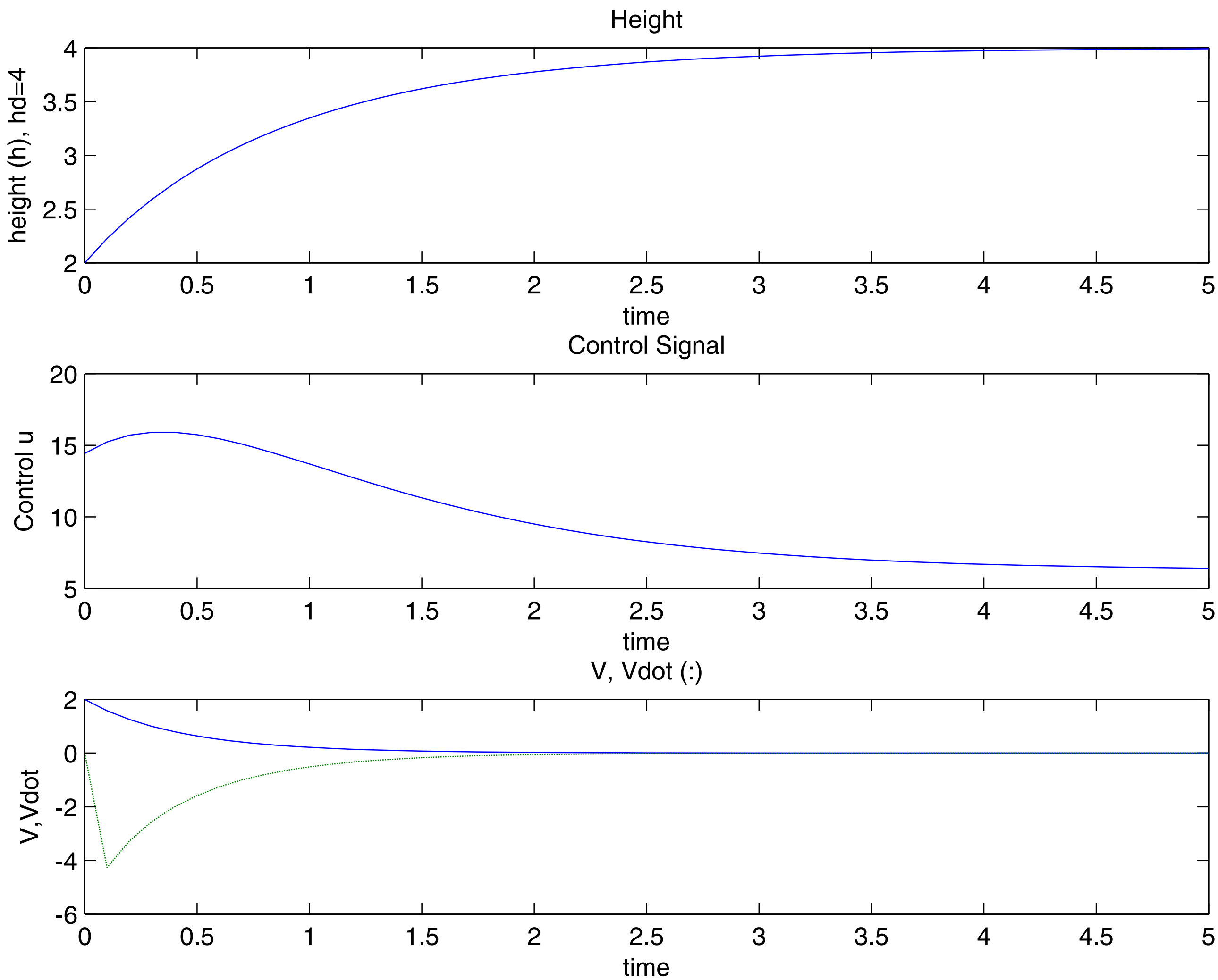
$\therefore$ all signals are bounded.

* Note that we have to apply u to h system into the h system: not the $\bar{h}$ system, transform u

$$u = a\sqrt{gh} - (h-h_d)\left[h^2+1\right]$$



Height - No Control Input (u=0)
height (h)
time
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5



Height
height (h), hd=4
time
Control Signal
Control u
time
V, Vdot (:)
V,Vdot
time

3. $\dot{x}_1 = -\cos(x_1) - x_1^3 + x_2 + 3 + x_{2d} - x_{2d}$

$\quad = -\cos(x_1) - x_1^3 + x_{2d} + 3 + \eta_2 \quad$ where $\eta_2 = x_2 - x_{2d}$

$V_1 = \frac{1}{2} x_1^2$

$\dot{V}_1 = x_1 \dot{x}_1 = x_1(-\cos(x_1) - x_1^3 + 3 + x_{2d}) + x_1 \eta_2$

let $x_{2d} = \cos(x_1) - 3$; $\dot{x}_{2d} = -\sin(x_1)\dot{x}_1$

$\quad = -\sin(x_1)(-\cos(x_1) - x_1^3 + x_2 + 3)$

$\dot{V}_1 = -x_1^4 + x_1 \eta_2$

$V = V_1 + \frac{1}{2}\eta_2^2$

$\dot{V} = \dot{V}_1 + \eta_2 \dot{\eta}_2 = \dot{V}_1 + \eta_2(\dot{x}_2 - \dot{x}_{2d})$

$\quad = -x_1^4 + \underline{x_1 \eta_2} + \eta_2\left(x_1 x_2 - u + \sin(x_1)(-\cos(x_1) - x_1^3 + x_2 + 3)\right)$

design $u = x_1 + x_1 x_2 + \eta_2 + \sin(x_1)(-\cos(x_1) - x_1^3 + x_2 + 3)$

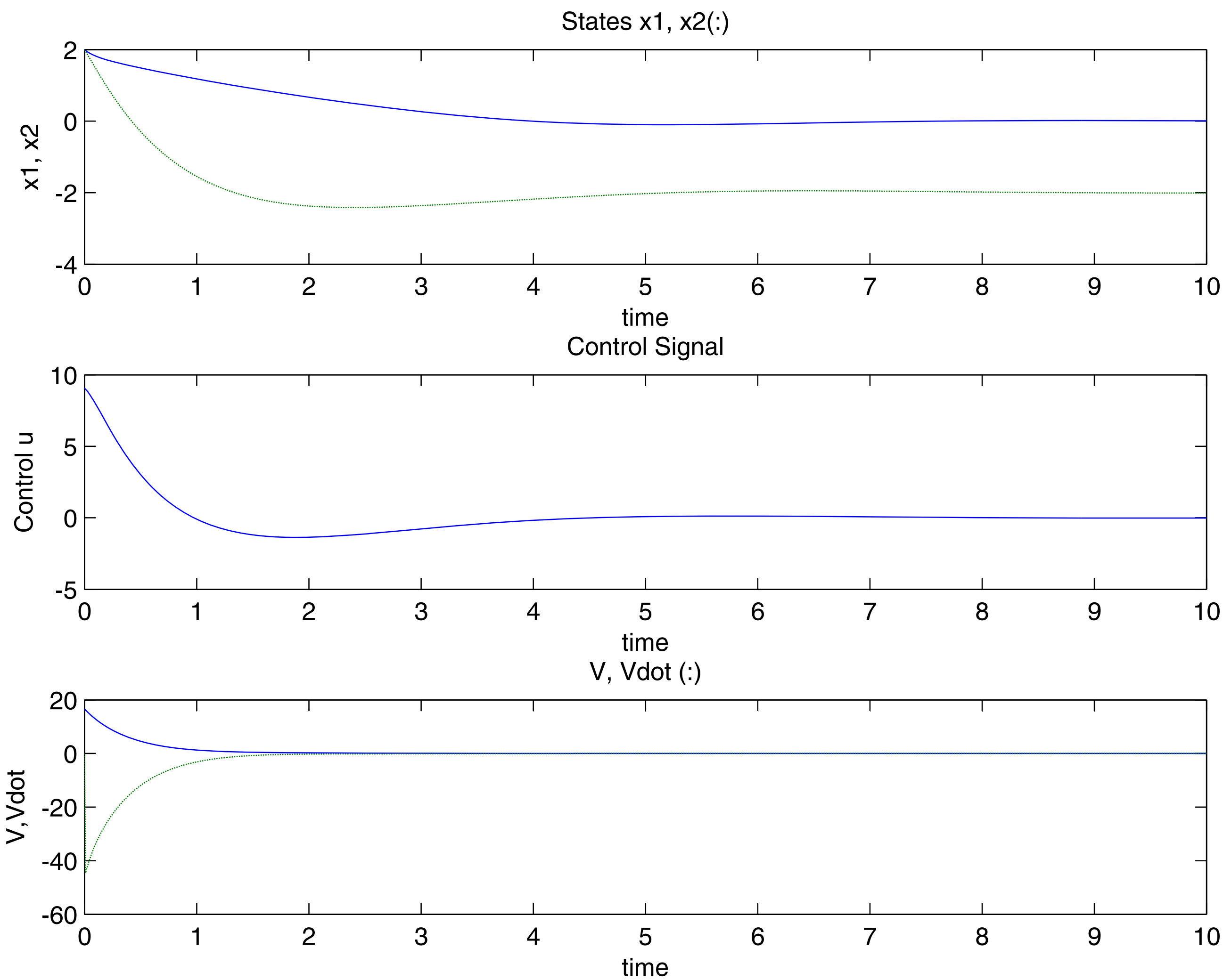
$\dot{V} = -x_1^4 - \eta_2^2$

V is PD, $V(0) = 0$, $\dot{V}$ is ND $\Rightarrow x_1, \eta_2 \to 0$

$x_1 \to 0 \Rightarrow x_{2d} = \cos(0) + 3 = 4$ i.e. $x_{2d} \to 4$ (bounded)

$\eta_2 \to 0, x_{2d} \to 4 \Rightarrow x_2 = \eta_2 + x_{2d} \to 0 + 4 = 4$ (bounded)

$x_1, \eta_2 \to 0, x_2 \to 4 \Rightarrow u \to 0$


States x1, x2(:)
x1, x2
2
0
-2
-4
0
1
2
3
4
5
6
7
8
9
10
time
Control Signal
Control u
10
5
0
-5
0
1
2
3
4
5
6
7
8
9
10
time
V, Vdot (:)
V,Vdot
20
0
-20
-40
-60
0
1
2
3
4
5
6
7
8
9
10
time

ECE 874 **Spring 2011 Test 2** Name ______________________

(100 points total)

1. (25 pts) Given the following system:

$$\dot{x}_1 = -2x_2$$

$$\dot{x}_2 \;=\; \mathrm{a}x_1 - \frac{1}{3}x_1^3 - 3x_2 + u$$

where a is an unknown constant. Design a tracking controller u so that the state x_1 follows x_{1d}.

Assume that the desired trajectory and the first two derivatives exist and are bounded.

Prove that the controller will work and that all signals remain bounded.

Solution:

Tracking in upper subsytem:

$e_1 = x_{1d} - x_1$

$\dot{e}_1 = \dot{x}_{1d} - \dot{x}_1 = \dot{x}_{1d} + 2x_2$

Introduce the embedded control:

$\dot{e}_1 = \dot{x}_{1d} + 2\eta_2 + 2x_{2d}$ where $\eta_2 = x_2 - x_{2d}$

Design "control input" x_{2d} :

$V_1 = \frac{1}{2}e_1{}^2$

$\dot{V}_1 = e_1\dot{e}_1 = e_1(\dot{x}_{1d} + 2\eta_2 + 2x_{2d})$

Design $x_{2d} = \frac{1}{2}\left(-\dot{x}_{1d} - k_1 e_1\right)$

$\dot{V}_1 = e_1\dot{e}_1 = -k_1 e_1^2 + 2e_1\eta_2$

$$\dot{\eta}_2 = \dot{x}_2 - \dot{x}_{2d} = \mathrm{a}x_1 - \frac{1}{3}x_1^3 - 3x_2 + u - \left(-\ddot{x}_{1d} - k_1\dot{e}_1\right)$$

$$\dot{\eta}_2 = \mathrm{a}x_1 - \frac{1}{3}x_1^3 - 3x_2 + \ddot{x}_{1d} + k_1\left(\dot{x}_{1d} + 2x_2\right) + u$$

$V_2 = V_1 + \eta_2^2$

$$\dot{V}_2 = \dot{V}_1 + \eta_2\dot{\eta}_2 = -k_1 e_1^2 + 2e_1\eta_2 + \eta_2\left(\mathrm{a}x_1 - \frac{1}{3}x_1^3 - 3x_2 + \ddot{x}_{1d} + k_1\left(\dot{x}_{1d} + 2x_2\right) + u\right)$$

Design $u = -\left[-\frac{1}{3}x_1^3 - 3x_2 + \ddot{x}_{1d} + k_1\left(\dot{x}_{1d} + 2x_2\right) + \hat{\mathrm{a}}x_1\right] - 2e_1 - \eta_2$

$\dot{V}_2 = -k_1 e_1^2 - \eta_2^2 + \eta_2\left(\mathrm{a}x_1 \text{-} \hat{\mathrm{a}}x_1\right)$

$V_3 = V_2 + \frac{1}{2}\tilde{a}^2$ where $\tilde{a} = a - \hat{a}$

$\dot{V}_3 = V_2 - \tilde{a}\dot{\hat{a}}$

$\dot{V}_3 = -k_1 e_1^2 - \eta_2^2 + \eta_2\tilde{a}x_1 - \tilde{a}\dot{\hat{a}}$

$\dot{V}_3 = -k_1 e_1^2 - \eta_2^2 + \tilde{a}\left(\eta_2 x_1 - \dot{\hat{a}}\right)$

Design $\dot{\hat{a}}_2 = \eta_2 x_1$

$\dot{V}_2 = -k_1 e_1^2 - \eta_2^2 \Rightarrow$ signals are bounded

$\ddot{V}_2 = -e_1\dot{e} - \eta_2\dot{\eta}_2 \Rightarrow \ddot{V}_2$ is bounded $\Rightarrow e_1 \to 0$

2. (25 pts) Design a tracking controller, $u(t)$, for the system:

$$\ddot{x} = 3\dot{x} + f(x) + 5 + u$$

where $f(x)$ is an unknown function.

Assume that the desired trajectory, x_d, and the first two derivatives exist and are bounded.

Prove that the controller will work and that all signals remain bounded.

Use the Lyapunov function candidate $V = \frac{1}{2}e^2 + \frac{1}{2}r^2$ where $e = x_d - x$ and $r = \dot{e} + \alpha e$.

When designing your control (not implementing your control) assume $f(x) = 2\sin(x)$

Find a bound for the "unknown" functions

$|f(x)| = |2\sin(x)| \le |2||\sin(x)| \le 2 = \rho(x)$

Use filtered tracking error

$\dot{r} = \ddot{e} + \alpha\dot{e} = \ddot{x}_d - \ddot{x} + \alpha\dot{e}$

$= \ddot{x}_d - (3\dot{x} + f(x) + 5 + u) + \alpha\dot{e}$

$V = \frac{1}{2}e^2 + \frac{1}{2}r^2$

$\dot{V} = e\dot{e} + r\dot{r} = -\alpha e^2 + er + r\left(\ddot{x}_d - (3\dot{x} + f(x) + 5 + u) + \alpha\dot{e}\right)$

$\dot{V} = -\alpha e^2 + er + r\left(\ddot{x}_d - 3\dot{x} - f(x) - 5 - u + \alpha\dot{e}\right)$

Design $u = \ddot{x}_d - 3\dot{x} - 5 + \alpha\dot{e} + e + V_{R1} + kr$

$\dot{V} = -\alpha e^2 - kr^2 + r\left(-f(x) - V_{R1}\right)$

$\dot{V} \le -\alpha e^2 - kr^2 + |r||f(x)| - rV_{R1}$; let $V_{R1} = \dfrac{r}{|r|}\rho(x)$

$\dot{V} \le -\alpha e^2 - kr^2 + |r|\rho(x) - \dfrac{r^2}{|r|}\rho(x) \Rightarrow$ used assumption 5 here! $\left(|f(x)| < \rho(x)\right)$

$\dot{V} \le -\alpha e^2 - kr^2 + |r|\rho(x) - |r|\rho(x) = -\alpha e^2 - kr^2$

$e^2 + r^2 = 2V$

$\dot{V} \le -2kV \Rightarrow \dot{V} + 2kV \le 0$

$\dot{V} + 2kV = -s(t)$, where $s(t) \ge 0$

$$V(t) = V(0)\exp(-2kt) - \exp(-2kt)\int_0^t \exp(2k\tau)s(\tau)d\tau$$

$V(t) \le V(0)\exp(-2kt)$

$\frac{1}{2}\left(e^2(t) + r^2(t)\right) \le \frac{1}{2}\left(e^2(t) + r^2(t)\right)\exp(-2kt)$

$\sqrt{\left|\left(e^2(t) + r^2(t)\right)\right|} \le \sqrt{\left|e^2(0) + r^2(0)\right|}\exp(-kt)$

3. (25 pts) Design a singularity free tracking controller, $u(t)$, for the system:

$$\left(x^2+1\right)\dot{x}=-x+u$$

Assume that the desired trajectory, x_d, and the first two derivatives exist and are bounded.
Prove that the controller will work and that all signals remain bounded.

Rewrite system as $\dot{x}=-\dfrac{x}{(x^2+1)}+\dfrac{1}{(x^2+1)}u$

$e_1 = x_{1d} - x_1$

$\dot{e}_1 = \dot{x}_{1d} - \dot{x}_1 = \dot{x}_{1d} - \left(-\dfrac{x}{(x^2+1)}+\dfrac{1}{(x^2+1)}u\right)$

$V_1 = \tfrac{1}{2}e_1^{\,2}$

$\dot{V}_1 = e_1\dot{e}_1 = e_1(\dot{x}_{1d} + \dfrac{x}{(x^2+1)} - \dfrac{1}{(x^2+1)}u)$

$u = x + (x^2+1)\left(\dot{x}_{1d} + ke_1\right)$

$\dot{V}_1 = -ke_1^2$

If you use in the control
$u = \ldots\ \dot{x}$ then you can't
show that u is bounded since $\dot{x} = \ldots\ u$

4. (25 pts) Design a tracking controller, $u(t)$, for the system:

$$\dot{x}_1 = -x_1^3 - 2x_1 + \sin(x_1 + x_1^2) - \frac{1}{3}u$$

Assume that the desired trajectory, x_{1d}, and the first two derivatives exist and are bounded.
a.) Prove that the controller will work and that all signals remain bounded.
b.) Simulate using Simulink using $x_{1d} = \cos(t)$.
Turn in plots of the state $x_1(t)$ and the control u(t) and your Simulink diagram.
Tracking in upper subsytem:

$e_1 = x_{1d} - x_1$

$\dot{e}_1 = \dot{x}_{1d} - \dot{x}_1 = \dot{x}_{1d} - \left(-x_1^3 - 2x_1 + \sin(x_1 + x_1^2) - \dfrac{1}{3}u\right)$

$V_1 = \tfrac{1}{2}e_1^{\,2}$

$\dot{V}_1 = e_1\dot{e}_1 = e_1(\dot{x}_{1d} + x_1^3 + 2x_1 - \sin(x_1 + x_1^2) + \dfrac{1}{3}u)$

$u = 3\left(-\dot{x}_{1d} - x_1^3 - 2x_1 + \sin(x_1 + x_1^2) - ke_1\right)$

$\dot{V}_1 = -ke_1^2$

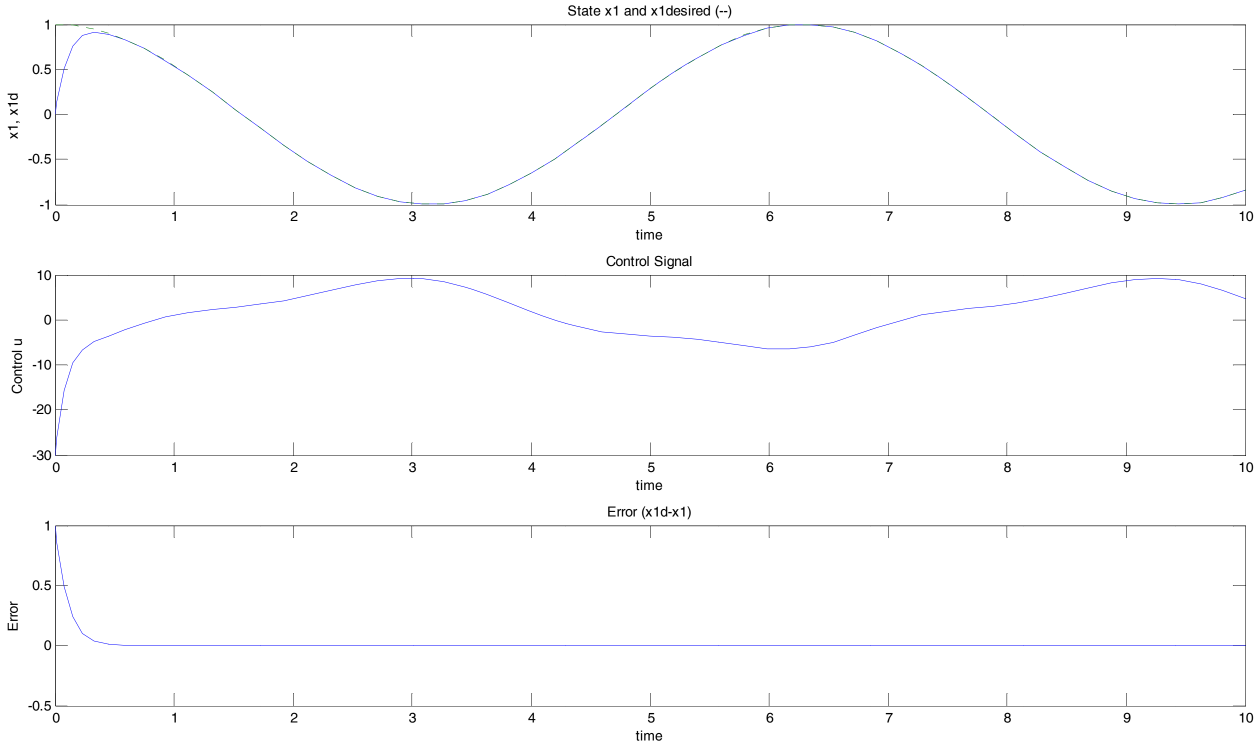
V_1 PD and $\dot{V}_1$ ND $\Rightarrow e_1 \to 0$

$e_1 \to 0, x_{1d}$ bounded $\Rightarrow x$ is bounded

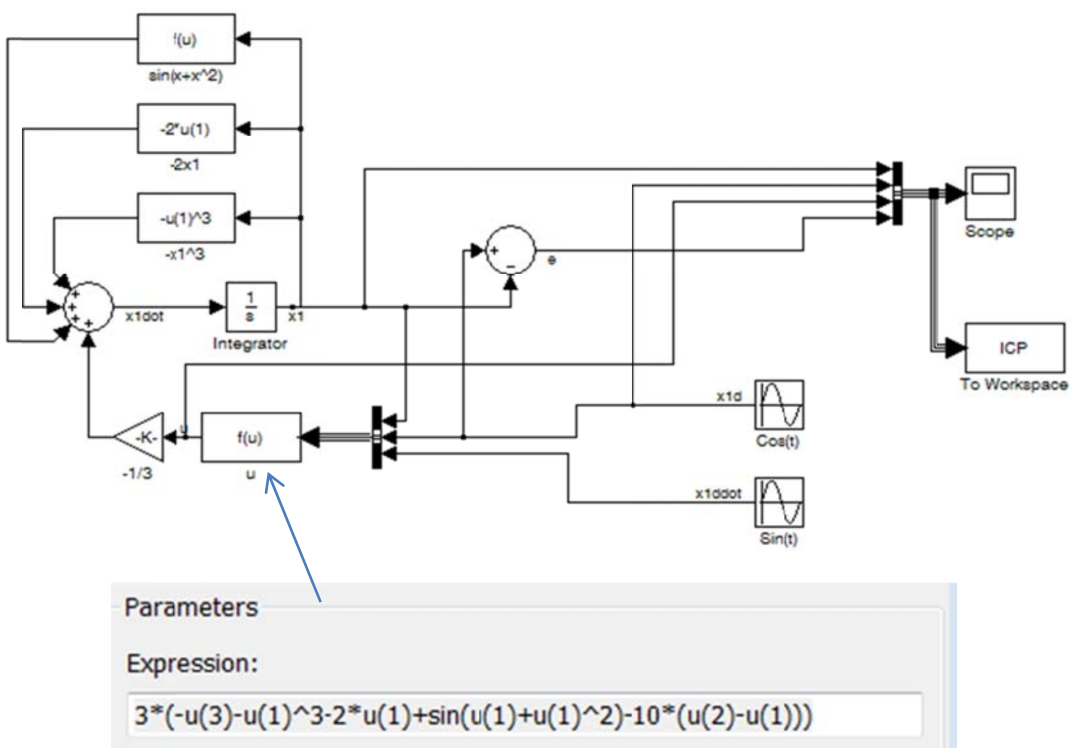
$e_1, x, \dot{x}_{1d}$ bounded $\Rightarrow u$ is bounded

x, u bounded $\Rightarrow \dot{x}_1$ is bounded

K=1



State x1 and x1desired (--)
x1, x1d
time
Control Signal
Control u
time
Error (x1d-x1)
Error
time



f(u)
sin(x+x^2)
-2*u(1)
-2x1
-u(1)^3
-x1^3
x1dot
1/s
x1
Integrator
e
Scope
ICP
To Workspace
x1d
Cos(t)
x1ddot
Sin(t)
-K-
-1/3
u
f(u)
u
Parameters
Expression:
3*(-u(3)-u(1)^3-2*u(1)+sin(u(1)+u(1)^2)-10*(u(2)-u(1)))

ECE 874 **Spring 2011 Test 3** Name ______________________

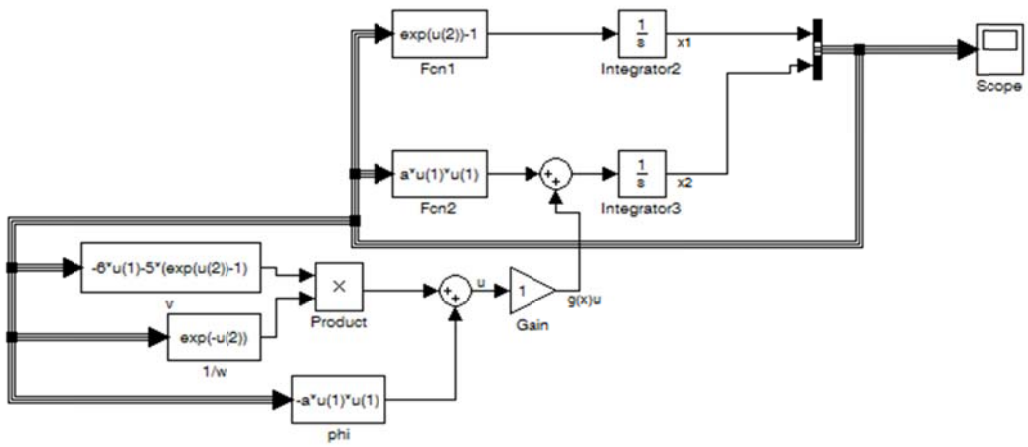
(100 points total)

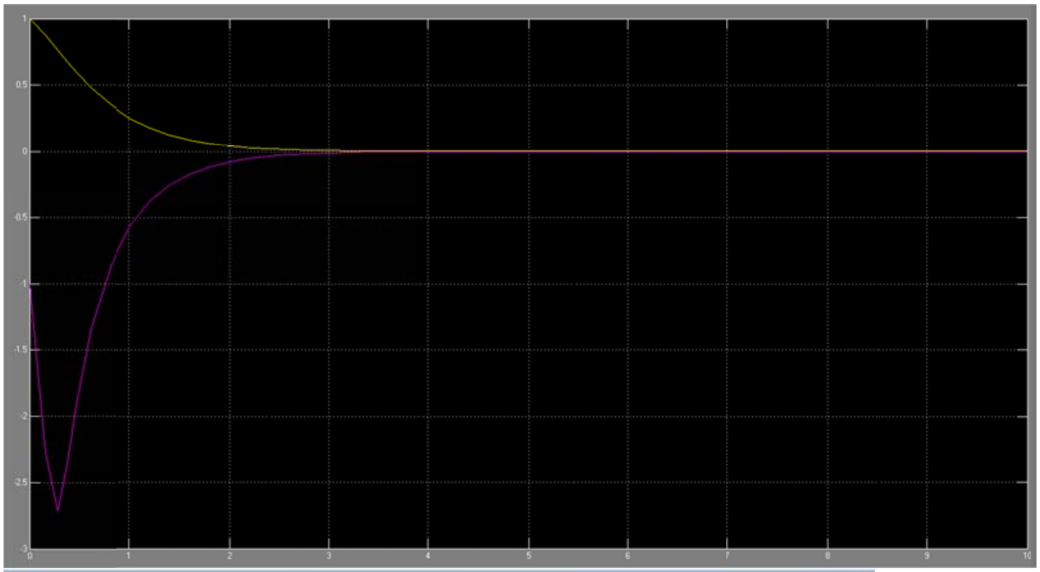
1. (25 pts) Start with Example10.10 in the Marquez book using $a = 1$.

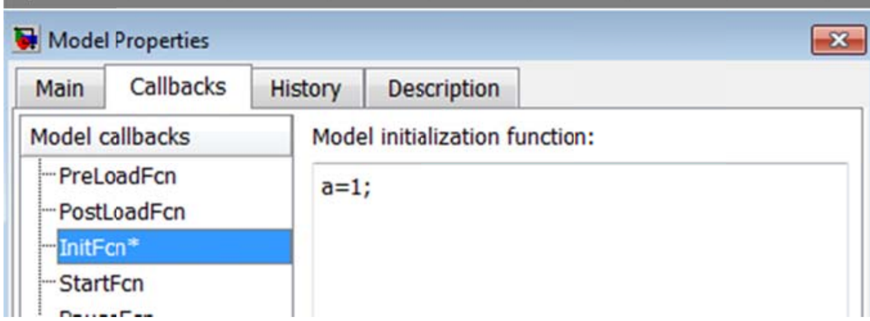
a.) Place the eigenvalues of the linearized system at -2 and -3 (use place() in MATLAB).

b.) Simulate the **linear system, ie z dynamics,** to show that *place*() has worked.

c.) Simulate the **entire system response** using the linearizing control and initial conditions $x(0)=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.






Model Properties
Main
Callbacks
History
Description
Model callbacks
PreLoadFcn
PostLoadFcn
InitFcn*
StartFcn
Model initialization function:
a=1;

2. (25 pts) Find the transformation $z = T(x)$ needed to design a control input u that will input-state linearize the system:

$$\dot{x}_1 = x_1 + x_2$$
$$\dot{x}_2 = x_3^2 + u$$
$$\dot{x}_3 = x_1 - x_2 + x_3$$

$$f(x) = \begin{bmatrix} x_1 + x_2 \\ x_3^2 \\ x_1 - x_2 + x_3 \end{bmatrix};\ g(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\frac{\partial T_1}{\partial x} g(x) = \frac{\partial T_1}{\partial x_2}(1) = 0 \Rightarrow T_1 \text{ is independent of } x_2$$

$$\frac{\partial T_2}{\partial x} g(x) = \frac{\partial T_2}{\partial x_2}(1) = 0 \Rightarrow T_2 \text{ is independent of } x_2$$

$$\frac{\partial T_3}{\partial x} g(x) = \frac{\partial T_3}{\partial x_2}(1) \neq 0 \Rightarrow T_3 \text{ must contain } x_2$$

$$\frac{\partial T_1}{\partial x} f(x) = \frac{\partial T_1}{\partial x_1}(x_1 + x_2) + \frac{\partial T_1}{\partial x_2}(x_3^2) + \frac{\partial T_1}{\partial x_3}(x_1 - x_2 + x_3) = T_2$$

$$\frac{\partial T_2}{\partial x} f(x) = \frac{\partial T_2}{\partial x_1}(x_1 + x_2) + \frac{\partial T_2}{\partial x_2}(x_3^2) + \frac{\partial T_2}{\partial x_3}(x_1 - x_2 + x_3) = T_3$$

Choose $\frac{\partial T_1}{\partial x_1} = 1$ and $\frac{\partial T_1}{\partial x_3} = 1$ then:

$$\frac{\partial T_1}{\partial x} f(x) = 1(x_1 + x_2) + 0 + 1(x_1 - x_2 + x_3) = 2x_1 + x_3 = T_2$$

choice $\Rightarrow$ $T_1 = x_1 + x_3$

$$\frac{\partial T_2}{\partial x} f(x) = (2)(x_1 + x_2) + 0 + (1)(x_1 - x_2 + x_3) = T_3$$

$3x_1 + x_2 + x_3 = T_3$

Check: $\frac{\partial T_3}{\partial x_2}(1) = 1(1) \neq 0$

3. (25 pts) Design an observer for $\dot{x}$ in the open-loop system

$$\ddot{x} = a\sin(x) + bx + c\cos(x) + u$$

where a,b, and c are known constants.

a.) Prove the theoretical performance of your design using a Lyapunov analysis.

b.) Show the observer in a form that could be implemented.

a. Design an observer to estimate $\dot{x}$ in the $\underbrace{\text{open-loop}}_{u=0}$ system:

$$\ddot{x} = a\sin(x) + bx + c\cos(x) + u$$

(x is measureable but $\dot{x}$ is not).

Define:

$$\tilde{x} = x - \hat{x}$$

$s = \dot{\tilde{x}} + \alpha\tilde{x}$ (similar to the filtered tracking error r) then $\dot{s} = \ddot{\tilde{x}} + \alpha\dot{\tilde{x}} = \ddot{x} - \ddot{\hat{x}} + \alpha\dot{\tilde{x}}$

propose $V = \frac{1}{2}\tilde{x}^2 + \frac{1}{2}s^2$

$$\dot{V} = \tilde{x}\dot{\tilde{x}} + s\dot{s} = \tilde{x}\dot{\tilde{x}} + s\left(\ddot{x} - \ddot{\hat{x}} + \alpha\dot{\tilde{x}}\right)$$

rearrange definition of s: $\dot{\tilde{x}} = s - \alpha\tilde{x}$

$$\dot{V} = \tilde{x}\left(s - \alpha\tilde{x}\right) + s\left(\ddot{x} - \ddot{\hat{x}} + \alpha\dot{\tilde{x}}\right)$$

$$= -\alpha\tilde{x}^2 + s\tilde{x} + s\left(\ddot{x} - \ddot{\hat{x}} + \alpha\dot{\tilde{x}}\right)$$

substitute the open-loop system ($\ddot{x}$ with $u = 0$):

$$\dot{V} = -\alpha\tilde{x}^2 + s\tilde{x} + s\left(a\sin(x) + bx + c\cos(x) + u - \ddot{\hat{x}} + \alpha\dot{\tilde{x}}\right)$$

we would like to have only $-\alpha\tilde{x}^2$ and -s^2 in $\dot{V}$, design $\ddot{\hat{x}}$ to make this happen:

$$\ddot{\hat{x}} = \underbrace{a\sin(x) + bx + c\cos(x) + \alpha\dot{\tilde{x}}}_{cancel} + \underbrace{s}_{stabilize} + \underbrace{\tilde{x}}_{cancel\ cross\ term}$$

$$\dot{V} = -\alpha\tilde{x}^2 - s^2$$

V is PD and radially unbounded, $\dot{V}$ is ND

$\Rightarrow \tilde{x}, s \to 0 \Rightarrow \hat{x} \to x$

$\Rightarrow \dot{\tilde{x}} = s - \alpha\tilde{x} \to 0 \Rightarrow \dot{\hat{x}} \to \dot{x}$

$\Rightarrow$ observer is bounded if $x, \dot{x}$ are bounded *b*.) Put observer into an implementable form.

$\ddot{\hat{x}} = a\sin(x) + bx + c\cos(x) + \alpha\dot{\tilde{x}} + s + \tilde{x}$

Two-part implementation of the filter:

$\dot{\hat{x}} = p + $(terms to get differentiated to make $\ddot{\hat{x}}$)

$\dot{p} =$ terms that don't get differentiated to make $\ddot{\hat{x}}$

Rewrite the observer by replacing s=$\dot{\tilde{x}} + \alpha\tilde{x}$ and regrouping

$\ddot{\hat{x}} = a\sin(x) + bx + c\cos(x) + \dot{\tilde{x}} + \alpha\tilde{x} + \tilde{x} + \alpha\dot{\tilde{x}}$

$= \underbrace{a\sin(x) + bx + c\cos(x) + (\alpha+1)\tilde{x}}_{put\ in\ \dot{p}} + \underbrace{(\alpha+1)\dot{\tilde{x}}}_{put\ in\ \dot{\hat{x}}}$

Implementable observer:

$\dot{\hat{x}} = p + (\alpha+1)\tilde{x}$

$\dot{p} = a\sin(x) + bx + c\cos(x) + (\alpha+1)\tilde{x}$

Prove that it works:

$\ddot{\hat{x}} = \dot{p} + (\alpha+1)\dot{\tilde{x}} = a\sin(x) + bx + c\cos(x) + (\alpha+1)\tilde{x} + (\alpha+1)\dot{\tilde{x}}$

4. (25 pts) Design a tracking controller, $u(t)$, using a filtering approach for the system:

$$\ddot{x} = a\sin(x) + b\cos(x+\pi/2) + 2 + u$$

where $\dot{x}$ cannot be measured and a=2 and b=3.

Assume that the desired trajectory, x_{1d}, and the first two derivatives exist and are bounded.

a.) Prove that the controller will work and that all signals remain bounded.

b.) Simulate using Simulink for $x_{1d} = \cos(t)$. Turn in plots of the state $x_1(t)$ and the control u(t) and your Simulink diagram.

a.) Define:

$$e = x_d - x \;\Rightarrow \dot{e} = \dot{x}_d - \dot{x} \;\Rightarrow \ddot{e} = \ddot{x}_d - \ddot{x}$$

$$\eta = \dot{e} + e + e_f \;\Rightarrow\; \dot{e} = \eta - e - e_f$$

$$\dot{e}_f = -e_f - k\eta + e$$

$$\dot{\eta} = \ddot{e} + \dot{e} + \dot{e}_f = \ddot{x}_d - \ddot{x} + \dot{e} + \dot{e}_f$$

$$= \ddot{x}_d - a\sin(x) - b\cos(x+\pi/2) - 2 - u + \dot{e} + \dot{e}_f$$

Propose: $V = \frac{1}{2}e^2 + \frac{1}{2}e_f^2 + \frac{1}{2}\eta^2$

$$V = \frac{1}{2}e^2 + \frac{1}{2}e_f^2 + \frac{1}{2}\eta^2$$

$$\dot{V} = e\dot{e} + e_f\dot{e}_f + \eta\dot{\eta}$$

$$= e\left(\eta - e - e_f\right) + e_f\left(-e_f - k_n\eta + e\right) + \eta\dot{\eta} = -e^2 - e_f^2 + e\eta - ke_f\eta + \eta\dot{\eta}$$

$$= -e^2 - e_f^2 + e\eta - ke_f\eta + \eta\left(\ddot{x}_d - a\sin(x) - b\cos(x+\pi/2) - 2 - u + \dot{e} + \dot{e}_f\right)$$

$$= -e^2 - e_f^2 + e\eta - ke_f\eta + \eta\left(\ddot{x}_d - a\sin(x) - b\cos(x+\pi/2) - 2 - u\right)$$

$$+ \eta\left(\eta - e - e_f\right) + \eta\left(-e_f - k\eta + e\right)$$

$$= -e^2 - e_f^2 - (k-1)\eta^2 + \eta\left(\ddot{x}_d - a\sin(x) - b\cos(x+\pi/2) - 2 - u\right) + \eta\left(-(k+2)e_f + e\right)$$

Assume for now e_f is measureable:

$$\dot{V} = -e^2 - e_f^2 - (k-1)\eta^2 + \eta\left(\underbrace{\ddot{x}_d - a\sin(x) - b\cos(x+\pi/2) - 2}_{cancel} - u\right) + \eta\left(\underbrace{-(k+2)e_f + e}_{cancel}\right)$$

Design $u = \ddot{x}_d - a\sin(x) - b\cos(x+\pi/2) - 2 - (k+2)e_f + e$

$$\dot{V} = -e^2 - e_f^2 - (k-1)\eta^2$$

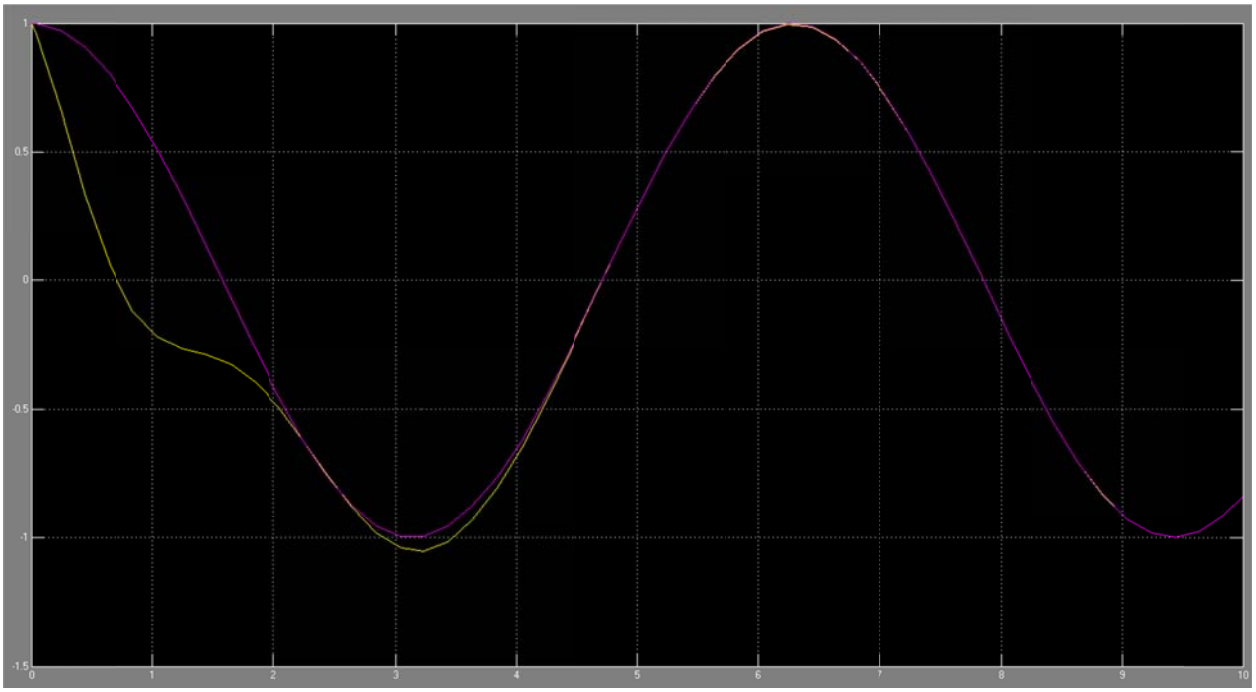
Choose $k > 1 \;\Rightarrow\;$ GES tracking b.) Define measureable form of e_f for the simulation:

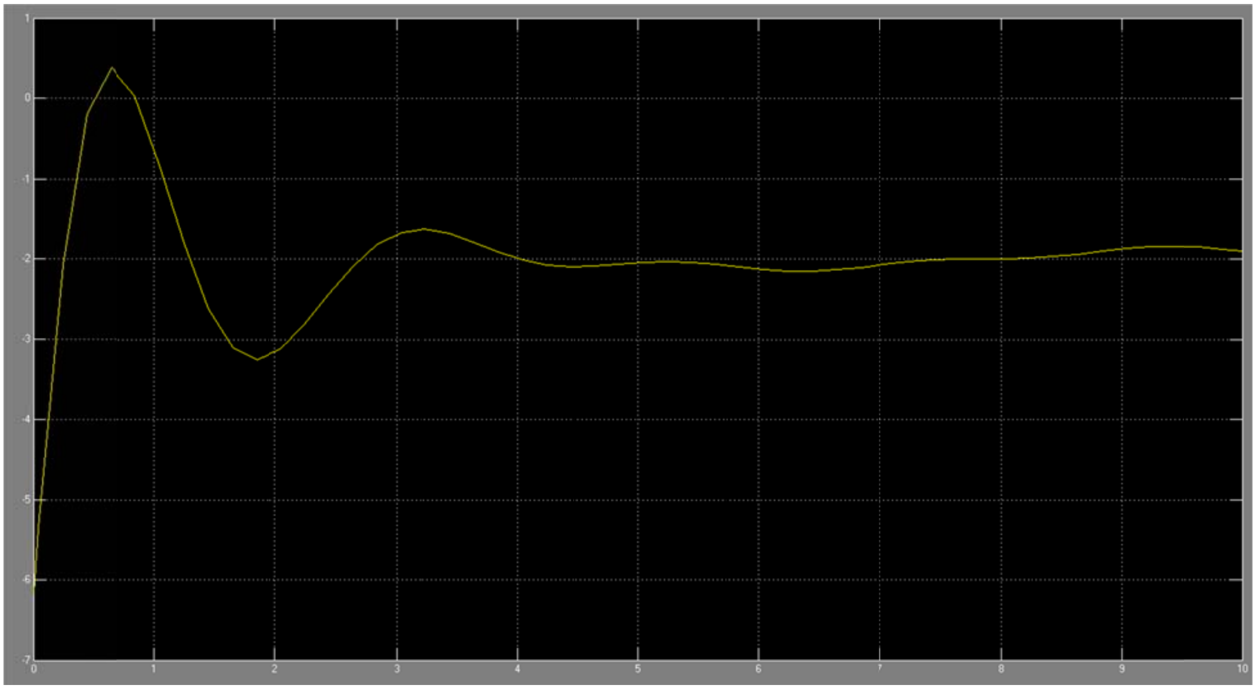
$$\dot{e}_f = -e_f - k\left(\dot{e} + e + e_f\right) + e = -(k+1)e_f - (k-1)e - k\dot{e}$$

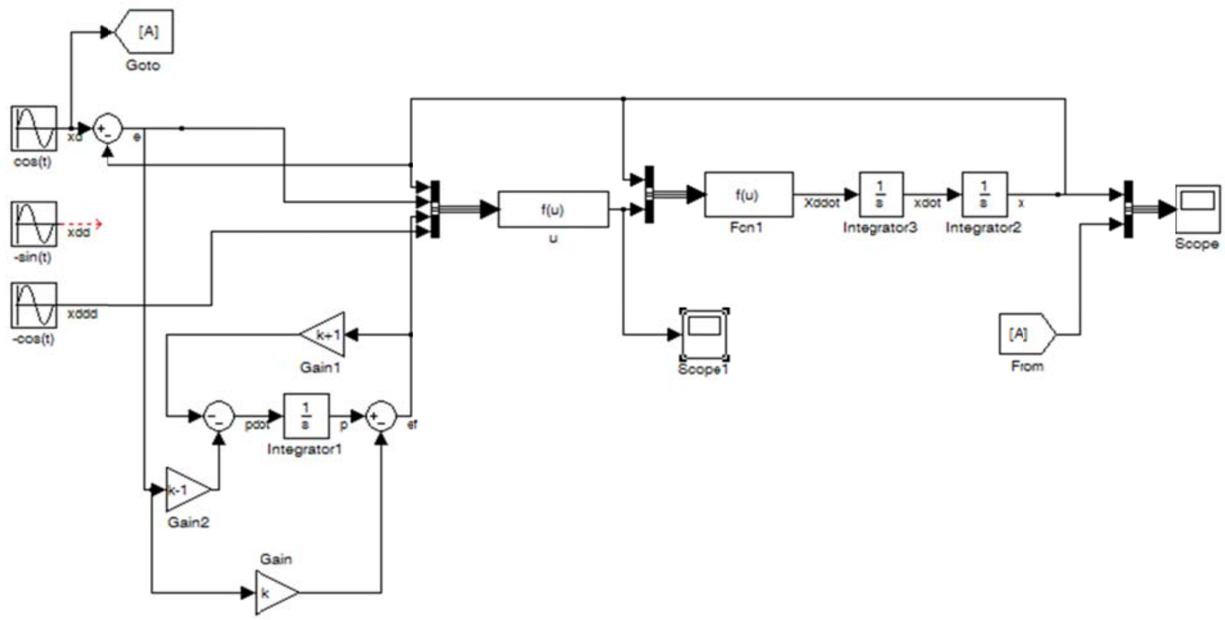
Two-part implementation of the filter:

$$e_f = p - ke$$

$$\dot{p} = -(k+1)e_f - (k-1)e$$







[A]
Goto
xd
e
cos(t)
xdd
-sin(t)
xddd
-cos(t)
f(u)
u
f(u)
Fcn1
Xddot
1
s
Integrator3
xdot
1
s
Integrator2
x
Scope
Scope1
[A]
From
k+1
Gain1
pdot
1
s
Integrator1
p
ef
k-1
Gain2
Gain
k